Student Number:____________________

Student Name ____________________

Teacher Name: ____________________

ABBOTSLEIGH

AUGUST 2011
YEAR 12
ASSESSMENT 4

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120
- Attempt Questions 1-8.
- All questions are of equal value.
- Answer each question in a new booklet.

Outcomes assessed

HSC course

- **E1** appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
- **E2** chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
- **E3** uses the relationship between algebraic and geometric representations of complex numbers and of conic sections
- **E4** uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
- **E5** uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces and resisted motion
- **E6** combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
- **E7** uses the techniques of slicing and cylindrical shells to determine volumes
- **E8** applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems
- **E9** communicates abstract ideas and relationships using appropriate notation and logical argument

Harder applications of the Extension 1 Mathematics course are included in this course. Thus the Outcomes from the Extension 1 Mathematics course are included.

From the Extension 1 Mathematics Course

Preliminary course

- **PE1** appreciates the role of mathematics in the solution of practical problems
- **PE2** uses multi-step deductive reasoning in a variety of contexts
- **PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- **PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- **PE5** determines derivatives that require the application of more than one rule of differentiation
- **PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

- **HE1** appreciates interrelationships between ideas drawn from different areas of mathematics
- **HE2** uses inductive reasoning in the construction of proofs
- **HE3** uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion or exponential growth and decay
- **HE4** uses the relationship between functions, inverse functions and their derivatives
- **HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- **HE6** determines integrals by reduction to a standard form through a given substitution
- **HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 Marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{1+x}{4+x^2}\,dx$ 2

(b) By completing the square find $\int \frac{1}{\sqrt{6-x^2-x}}\,dx$ 2

(c) Find $\int \sin^3 x \cos^3 x\,dx$ 2

(d) Use integration by parts to evaluate $\int_0^{\ln 2} xe^{-x}\,dx$. Give your answer in simplest form. 3

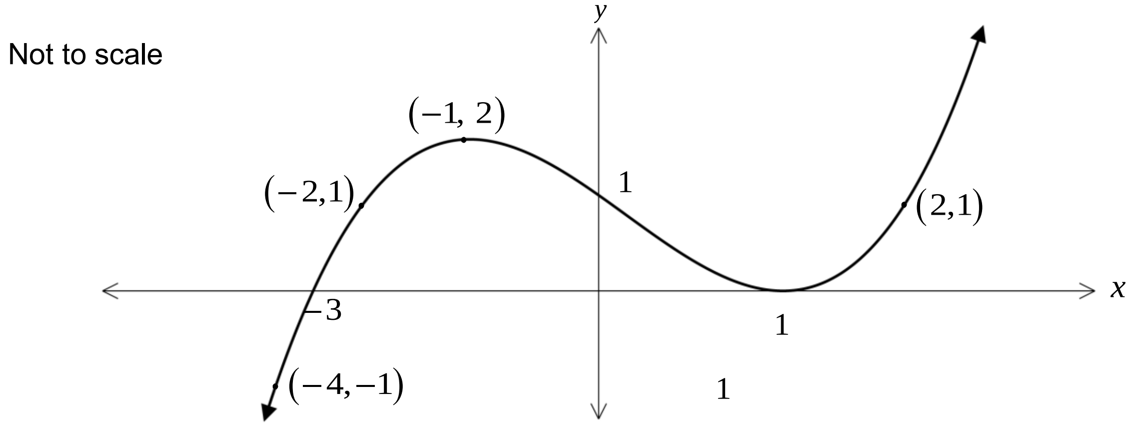
(e) By making the numerator rational, or otherwise, find $\int \sqrt{\frac{5-x}{5+x}}\,dx$ 3

(f) Use a trigonometric substitution to find $\int \frac{dx}{x^2\sqrt{4-x^2}}$ 3

Marks

QUESTION 2 (15 Marks) Use a SEPARATE writing booklet.

(a) The following is a sketch of a function $y = f(x)$



Draw separate one-third page sketches of the following: (clearly showing important features)

(i) $y = -f(x)$ 1

(ii) $y = \sqrt{f(x)}$ 1

(iii) $y = f(1-x)$ 2

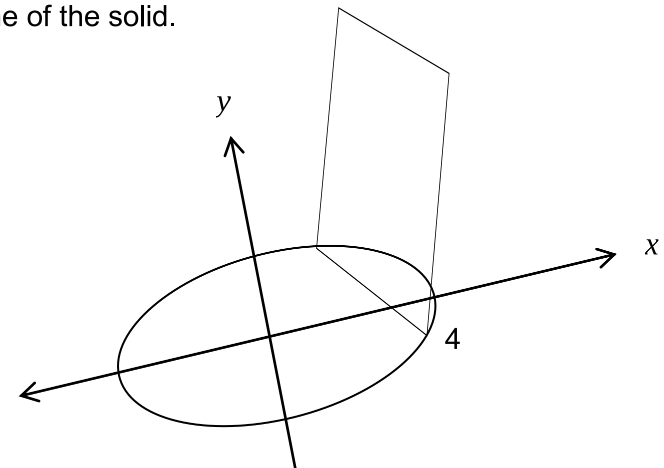
(iv) $y = \cos^{-1} f(x)$ 2

(v) $y = \dfrac{1}{1-f(x)}$ 2

(b) Write down the equation of $P(x)$ if it is a monic polynomial of degree 3 with integer coefficients, a constant term of 12 and one root equal to $\sqrt{3}$. Leave your answer in factored form. 2

(c) Evaluate $\int_0^3 |x+1|\, dx$ 2

(d) The base of a solid is in the circle $x^2 + y^2 = 16$ and every plane section perpendicular to the x axis is a rectangle whose height is twice its base (which lies inside the circle). Find the volume of the solid. 3



Marks

QUESTION 3 (15 Marks) Use a SEPARATE writing booklet.

(a) Let $z = \dfrac{2-3i}{1+i}$

(i) Find $\overline{z}$ in the form $x + iy$ 2

(ii) Evaluate $|z|$ 1

(b) Consider $w = -\sqrt{3} + i$

(i) Express w in modulus-argument form 2

(ii) Hence or otherwise show that $w^7 + 64w = 0$ 2

(c) Sketch the region in the complex plane where the inequalities $1 \le |z - i| \le 2$ and $\text{Im}(z) \ge 0$ hold simultaneously.
Clearly mark in all *x and y* intercepts. 3

(d) In an Argand diagram z is a point on the circle $|z| = 2$.

Given that $\arg z = \theta$ and $0 < \theta < \dfrac{\pi}{2}$

(i) Draw a diagram to represent this information. 1

(ii) Find, in terms of θ, an expression for $\arg z^2$ 1

(iii) Find, in terms of θ, giving brief reasons, expressions for :

(A) $\arg(z+2)$ 1

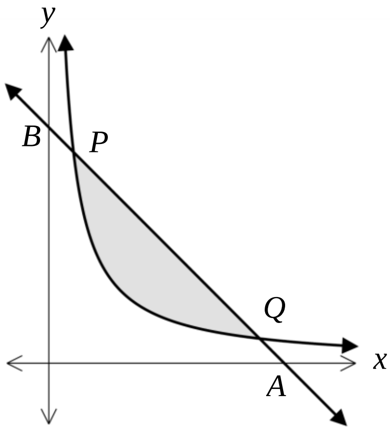
(B) $\arg(z-2)$ 1

(C) $\left|\dfrac{z-2}{z+2}\right|$ 1

Marks

QUESTION 4 (15 Marks) Use a SEPARATE writing booklet.

(a) Consider the rectangular hyperbola $xy = c^2$ where $c > 0$.



(i) Prove that the equation of the chord joining points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ where $0 < p < q$ is given by $x + pqy = c(p+q)$. 2

(ii) The chord PQ intersects the x and y axes at A and B respectively. Prove $AP = BQ$. 2

(iii) Show that the area enclosed by the hyperbola $xy = c^2$ and chord PQ is $\frac{c^2\left(q^2 - p^2\right)}{2pq} + c^2\ln\left(\frac{p}{q}\right)$ square units. 2

(b) (i) Divide the polynomial $P(x) = x^4 + 3x^3 - 7x^2 + 11x - 1$ by $x^2 + 2$ and write your result in the form $P(x) = \left(x^2 + 2\right)Q(x) + cx + d$. 2

(ii) Hence determine the values of a and b for which the polynomial $\left(x^4 + 3x^3 - 7x^2 + 2x\right) + ax + b$ is exactly divisible by $x^2 + 2$. 2

(c) The equation $|z - 3| + |z + 3| = 10$ corresponds to an ellipse in the Argand diagram.

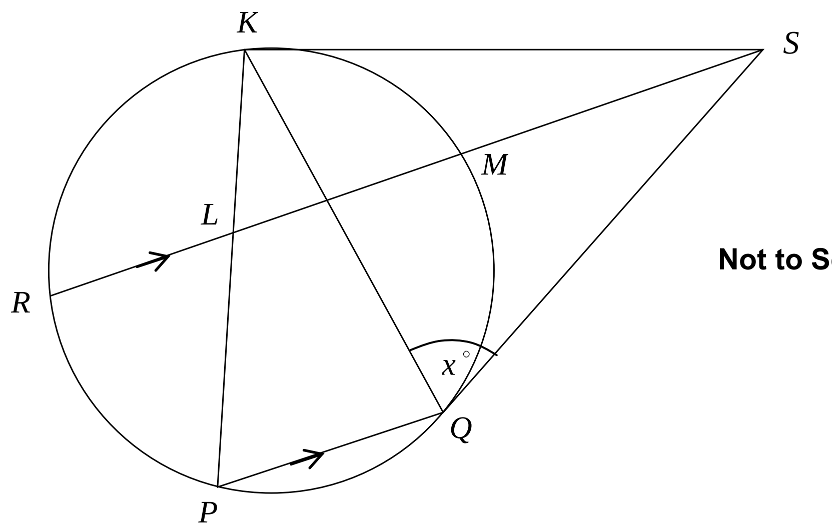
(i) Prove that the equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{16} = 1$ 3

(ii) Sketch the ellipse showing all important features. 2

QUESTION 5 (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) If $u_1 = 1$, $u_2 = 5$ and $u_n = 5u_{n-1} - 6u_{n-2}$ for integers $n \geq 3$, prove by induction that $u_n = 3^n - 2^n$ for integers $n \geq 1$. 3

(b) In the diagram below PQ and RM are parallel chords in a circle. The tangent at Q meets RM produced at S and SK is another tangent to the circle. PK cuts RM at L.



(i) Copy or trace this diagram into your answer booklet.
Let $\angle SQK = x^\circ$ and prove $\angle SQK = \angle SLK$ 2

(ii) Explain why $LKSQ$ is a cyclic quadrilateral. 1

(iii) Prove $PL = QL$ 3

(c) It is given that $\sum_{r=0}^{n} (-1)^n \dfrac{{}^nC_r}{x+n} = \dfrac{n!}{x(x+1)(x+2)......(x+n)}$. (DO NOT PROVE)

Hence prove $1 - \dfrac{1}{2}\,{}^nC_1 + \dfrac{1}{3}\,{}^nC_2 - ....\dfrac{(-1)^n\ {}^nC_n}{n+1} = \dfrac{1}{n+1}$ 2

Question 5 continues on the next page.

(d) The curve $y = 8x - x^2$ and the line $y = 12$ is sketched below.

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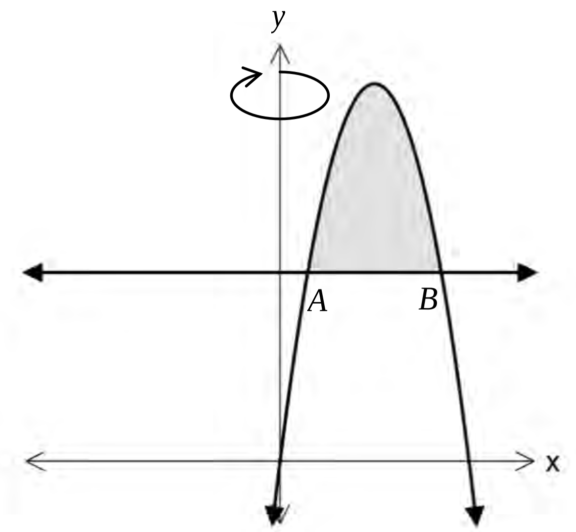


(i) Find the coordinates of the points of intersection A and B 1

(ii) The shaded area is rotated around the y axis.
Use the method of **cylindrical shells** to find the exact volume formed. (You may leave your answer **unsimplified** in fractional form) 3

Marks

QUESTION 6 (15 Marks) Use a SEPARATE writing booklet.

(a) (i) If $\dfrac{1}{x(\pi - 2x)} = \dfrac{A}{x} + \dfrac{B}{\pi - 2x}$ and $A = \dfrac{1}{\pi}$ find B in terms of π. 1

(ii) Hence show that $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{x(\pi - 2x)} = \frac{2}{\pi}\ln 2$ 3

(iii) By using the substitution $u = a + b - x$ show that

$\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$ 1

(iv) Hence evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x\,dx}{x(\pi - 2x)}$ 3

(b) A curve is defined by the equation $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$

(i) Show that $1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{y^2}{a^2}$ 3

(ii) The arc length S between points $(0, a)$ and (x, y) of the curve is given by

$S = \displaystyle\int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ (DO NOT PROVE THIS)

Show that $S = \sqrt{y^2 - a^2}$ 4

Marks

QUESTION 7 (15 Marks) Use a SEPARATE writing booklet.

(a) (i) Explain why the domain of the function, $f(x)=\sqrt{2-\sqrt{x}}$ is $0 \le x \le 4$ 1

(ii) Show that $f(x)$ is a decreasing function and hence find its range. 2

(iii) Using the substitution, $u = 2-\sqrt{x}$ or otherwise, find the area bounded by the curve and the x and y axes. 3

(b) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$ where n is an integer and $n \ge 3$.

Show that $I_n + I_{n-2} = \frac{1}{n-1}$ 3

(c) A body mass of 1 kg falls vertically downwards, from rest, in a medium which exerts a resistance to its motion of $\frac{1}{100}v^2$ Newtons (where v metres per second is the speed of the body when it has fallen a distance of x metres).

(i) Show (on a diagram) that the equation of motion of the body is $\ddot{x} = g - \frac{1}{100}v^2$ where g is the acceleration due to gravity. 1

(ii) Show that the terminal speed V_T is given by $V_T = 10\sqrt{g}$ 2

(iii) Prove that $v^2 = (V_T)^2\left(1 - e^{-\frac{x}{50}}\right)$ 3

QUESTION 8 (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) A curve is defined implicitly by the equation $x^2 + 2xy + y^5 = 4$

(i) Show that the gradient of the tangent at $P\left(X, Y\right)$ is given by

$$\frac{dy}{dx} = \frac{-2X - 2Y}{5Y^4 + 2X}$$ 2

(ii) The tangent is horizontal at P. Show that X satisfies $X^5 + X^2 + 4 = 0$. 1

(iii) Show that X is the unique real solution of $X^5 + X^2 + 4 = 0$ and that $-2 < X < -1$ 3

(b) (i) Solve $\tan 4\theta = 1$ for $0 \le \theta \le \pi$ 1

(ii) Express $\tan 2\theta$ in terms of $\tan\theta$. 1

(iii) Hence show $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$. 2

(iv) Hence show $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ has roots $\tan\frac{\pi}{16}$, $\tan\frac{5\pi}{16}$, $\tan\frac{9\pi}{16}$ and $\tan\frac{13\pi}{16}$. 2

(v) Hence evaluate $\tan\frac{\pi}{16}\tan\frac{5\pi}{16}\tan\frac{9\pi}{16}\tan\frac{13\pi}{16}$ 1

(vis1) By solving $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ another way, show the exact value of $\tan\frac{\pi}{16} - \cot\frac{\pi}{16} = -2 - 2\sqrt{2}$. 2

END OF PAPER

TABLE OF STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \quad = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

$NOTE: \quad \ln x = \log_e x, \qquad x > 0$

Ex 2 Trial Solutions 2011

Q 1

a) $\int \frac{1+x}{4+x^2}\,dx = \int \frac{1}{4+x^2}\,dx + \frac{1}{2}\int \frac{2x}{4+x^2}\,dx$

$= \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{1}{2}\ln(4+x^2) + c$

b) $\int \frac{1}{\sqrt{6-x^2-x}} = \int \frac{1}{\sqrt{-(x^2+x-6)}}$

$= \int \frac{1}{\sqrt{-\left[(x^2+x+\frac{1}{4}) - \frac{25}{4}\right]}}$

$= \int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2}}$

$= \sin^{-1}\frac{x+\frac{1}{2}}{\frac{5}{2}} + c$

$= \sin^{-1}\frac{2x+1}{5} + c$

c) $\int \sin^3 x \cos^3 x\,dx = \int \sin^3 x \cos x(1-\sin^2 x)\,dx$

$= \int \sin^3 x \cos x - \int \sin^5 x \cos x\,dx$

$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + c$

d) $\int_0^{\ln 2} x e^{-x}\,dx$ let $u = x$, $du = dx$, $dv = e^{-x}dx$, $v = -e^{-x}$

$= \left[-x e^{-x}\right]_0^{\ln 2} + \int_0^{\ln 2} e^{-x}\,dx$

$= -\ln 2\, e^{-\ln 2} - 0 + \left[-e^{-x}\right]_0^{\ln 2}$

$= -\ln 2 \times \frac{1}{2} + \left(-e^{-\ln 2} + e^0\right)$

$= -\frac{\ln 2}{2} - \frac{1}{2} + 1 = \frac{1-\ln 2}{2}$

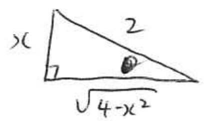
Q1 e) $\int \frac{\sqrt{5-x}}{\sqrt{5+x}} \cdot \frac{\sqrt{5-x}}{\sqrt{5-x}} = \int \frac{5-x}{\sqrt{25-x^2}}\,dx$

$= \int \frac{5}{\sqrt{25-x^2}}\,dx + \frac{1}{2}\int \frac{-2x}{\sqrt{25-x^2}}\,dx$

$= 5\sin^{-1}\frac{x}{5} + \sqrt{25-x^2} + c$

f) $I = \int \frac{dx}{x^2\sqrt{4-x^2}}$

Construct right angled Δ with smaller length $\sqrt{4-x^2}$:



$\Rightarrow \sin\theta = \frac{x}{2}$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

also, $\cos\theta = \frac{\sqrt{4-x^2}}{2}$

$\therefore \sqrt{4-x^2} = 2\cos\theta$

$\therefore I = \int \frac{2\cos\theta\,d\theta}{4\sin^2\theta \cdot 2\cos\theta}$

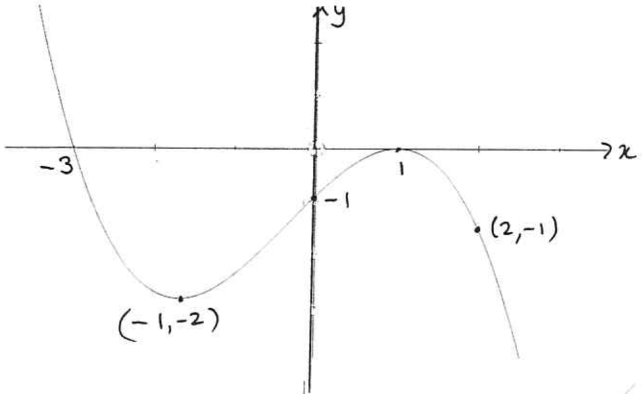
$= \frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$

$= \frac{1}{4}\int \csc^2\theta\,d\theta$

$= -\frac{1}{4}\cot\theta + c$

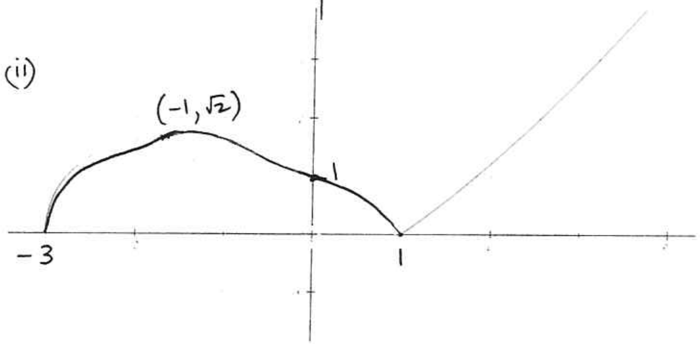
$= -\frac{\sqrt{4-x^2}}{4x} + c$

2) (a) (i) All diagrams are not strictly to scale.



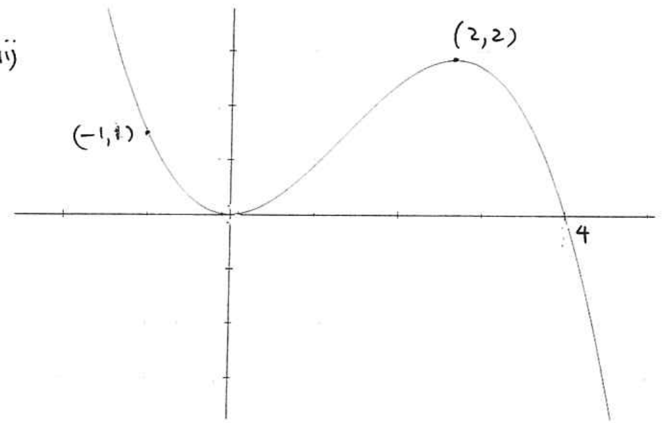
$y = -f(x)$

(ii)



$y = \sqrt{f(x)}$

(iii)



$y = f(1-x) = f(-(x-1))$

flip over y axis and shift 1 to the right

$x=0 \quad f(1) = 0$

$x=1 \quad f(0) = 1$

$x=2 \quad f(-1) = 2$

$x=-1 \quad f(2) = 1$

$x=3 \quad f(-2) = 1$

$x=4 \quad f(-3) = 0$

(iv)

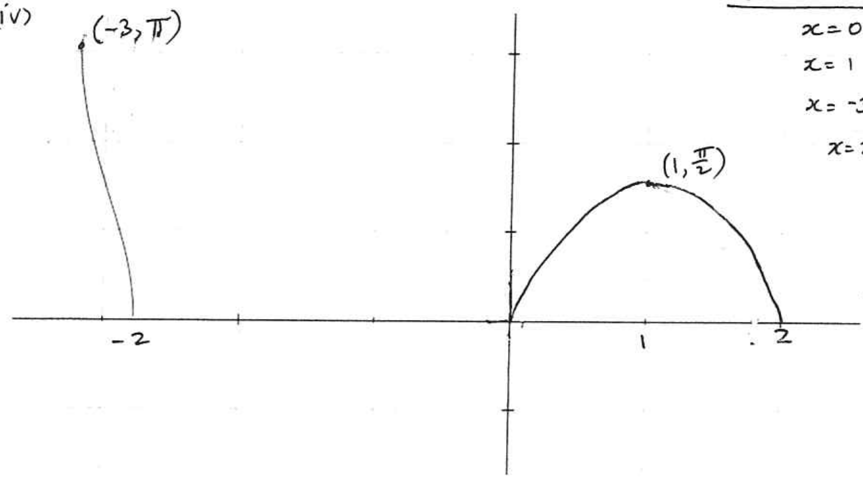
$y = \cos^{-1}(f(x))$

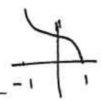
$x=0 \quad \cos^{-1} 1 = 0$

$x=1 \quad \cos^{-1} 0 = \frac{\pi}{2}$

$x=-3 \quad \cos^{-1} -1 = \pi$

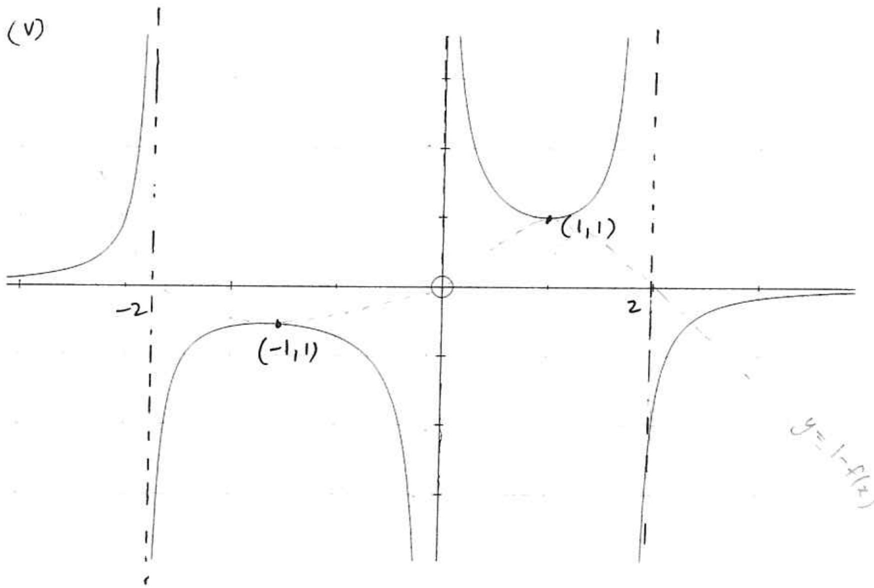
$x=2 \quad \cos^{-1} 1 = 0$





Gap between $x=-2$ and $x=0$ as $f(x) > 1$ [not defined for $\cos^{-1}$]

(v)

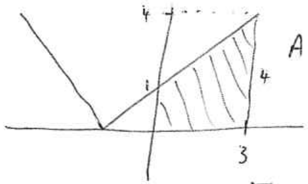


Sketch $y = 1 - f(x)$ (shift (i) up by 1 unit)

Then sketch reciprocal.

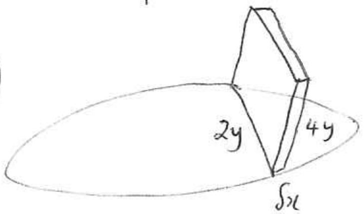
Q2b) $P(x) = (x^2-3)(x-4)$

c) $\int_0^3 |x+1|\,dx$ = area under graph:



$A = \frac{1}{2} \cdot 3\,(1+4)$
$= \frac{15}{2}$

(d)



typical rectangular prism has vol:
$= 8y^2\,\delta x$

$\therefore V = \int_{-4}^{4} 8y^2\,dx$

given the symmetry $V = 2\int_0^4 8y^2\,dx$

$= 16\int_0^4 16 - x^2\,dx$

$= 16\left[16x - \frac{x^3}{3}\right]_0^4$

$= 16\left[\left(64 - \frac{64}{3}\right) - 0\right]$

$= \frac{2048}{3}\,u^3$

Q3

(a)(i) $\frac{2-3i}{1+i} \cdot \frac{1-i}{1-i}$

$= \frac{2 - 2i - 3i + 3i^2}{1+i}$

$= -\frac{1}{2} - \frac{5i}{2}$

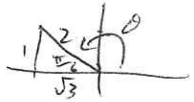
$\therefore \bar{Z} = -\frac{1}{2} + \frac{5i}{2}$

(ii) $|Z| = \sqrt{\frac{1}{4} + \frac{25}{4}}$

$= \frac{\sqrt{26}}{2}$

(b)(i) $\omega = -\sqrt{3} + i$

$\therefore \omega = 2\,\text{cis}\,\frac{5\pi}{6}$



(ii) $\omega^7 + 64\omega = 2^7 \text{cis}\,\frac{35\pi}{6} + 64 \cdot 2\,\text{cis}\,\frac{5\pi}{6}$

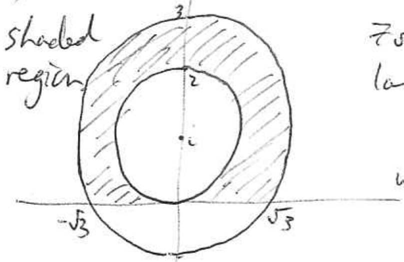
$= 128\,\text{cis}\left(-\frac{\pi}{6}\right) + 128\,\text{cis}\,\frac{5\pi}{6}$

$= -128\,\text{cis}\,\frac{5\pi}{6} + 128\,\text{cis}\,\frac{5\pi}{6}$

$= 0$

(c)

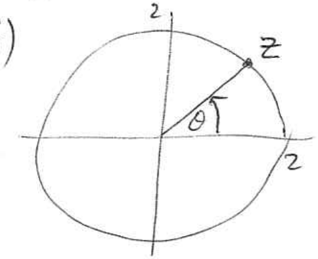
shaded region



For x intercepts, larger circle given by
$x^2 + (y-1)^2 = 4$
when $y = 0$
$x^2 + 1 = 4$
$x^2 = 3$
$x = \pm\sqrt{3}$

Q3 (d)

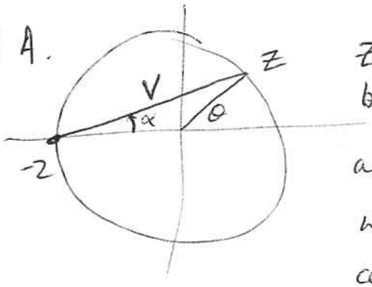
(i)



(ii) $\arg z^2 = 2\arg z$

$= 2\theta$

(iii) A.

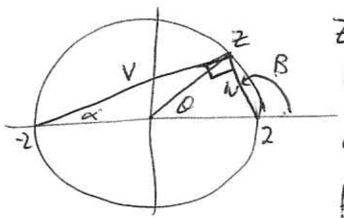


$z+2$ can be represented by the vector, V.

$\arg(z+2) = \alpha$

where $\alpha = \frac{\theta}{2}$ (angle at the centre = twice angle at the circumference)

B



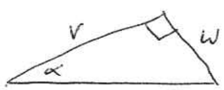
$z-2$ can be represented by the vector, w

$\arg(z-2) = \beta$

$\beta = \alpha + 90°$ (ext angle of Δ, the Δ is right-angled (angle in semi-circle))

$= \frac{\theta}{2} + 90$

C. $\left|\frac{z-2}{z+2}\right| = \frac{|z-2|}{|z+2|} = \frac{|w|}{|V|}$



Now $\tan\alpha = \frac{|w|}{|V|}$

i.e $\tan\frac{\theta}{2} = \frac{|w|}{|V|}$

Q4

a) $xy = c^2$

(i) $M_{PQ} = \dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \dfrac{c\,\frac{q-p}{pq}}{c(p-q)} = \dfrac{\frac{-(p-q)}{pq}}{(p-q)} = -\frac{1}{pq}$

$\therefore y - \frac{c}{p} = -\frac{1}{pq}(x - cp)$

$pqy - cq = -x + cp$

$x + pqy = c(p+q)$

(ii) Chord at x axis : $x = c(p+q)$ i.e $A[(c(p+q), 0)]$

at y axis : $pqy = c(p+q)$ i.e $B\left(0, \frac{c(p+q)}{pq}\right)$

$(d_{AP})^2 = (cp - c(p+q))^2 + \left(\frac{c}{p}\right)^2$

$= c^2q^2 + \frac{c^2}{p^2}$

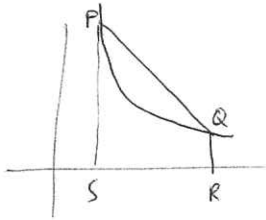
$(d_{BQ})^2 = (cq - 0)^2 + \left[\frac{c}{q} - \frac{c(p+q)}{pq}\right]^2$

$= c^2q^2 + \left[\frac{cp - cp - cq}{pq}\right]^2$

$= c^2q^2 + \frac{c^2}{p^2}$

$= (d_{AP})^2$

$\therefore d_{AP} = d_{BQ}$



(iii) Area = trapezium PQRS $- \int_{cp}^{cq} \frac{c^2}{x}\,dx$

$= \frac{1}{2} c(q-p)\left[c\left(\frac{1}{p} + \frac{1}{q}\right)\right] - c^2 \int_{cp}^{cq} \frac{1}{x}\,dx$

$= \frac{1}{2} c^2 (q-p)\frac{(q+p)}{pq} - c^2\left[\ln x\right]_{cp}^{cq}$

$= \frac{1}{2} c^2 \frac{(q^2-p^2)}{pq} - c^2(\ln cq - \ln cp)$

$= \frac{c^2(q^2-p^2)}{\ldots} + c^2 \ln\left(\frac{p}{q}\right)$

$\ln cq - \ln cp$

$= \ln\frac{cq}{cp}$

$= -\ln\frac{p}{q}$

4(b) (i)

$$\begin{array}{r} x^2 + 3x - 9 \\ x^2+2 \enclose{longdiv}{x^4 + 3x^3 - 7x^2 + 11x - 1} \\ -(x^4 + 2x^2) \\ 3x^3 - 9x^2 \\ -(3x^3 + 6x) \\ -9x^2 + 5x \\ -(-9x^2 - 18) \\ 5x + 17 \end{array}$$

$\therefore P(x) = (x^2+2)[x^2+3x-9] + 5x + 17$

(ii) x^2+2 divides into the first 3 terms as above, but the algorithym gets modified with the $2x$:

$$\begin{array}{r} 2x \\ -(\quad +6x) \\ -9x^2 - 4x \\ -(-9x^2 - 18) \\ -4x + 18 \end{array}$$
← Remainder.

$\therefore ax + b \equiv 4x - 18$ to ensure the remainder cancels.

alternatively: use result

$x^4 + 3x^3 - 7x^2 + 11x - 1 = (x^2+2)(x^2+3x-9) + 5x + 17$

$x^4 + 3x^3 - 7x^2 + 11x - 1 - 5x - 18 = (x^2+2)(x^2+3x-9)$

$x^4 + 3x^3 - 7x^2 + 6x - 18 = (x^2+2)(x^2+3x-9)$

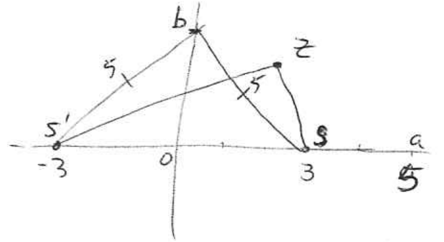
$b = -18$

$2x + ax = 6x$

$a = 4$

4(c)

(i) $|z-3| + |z+3| = 10$



$zS + zS' = 2a$ by definition.
$= 10$
$\therefore a = 5$

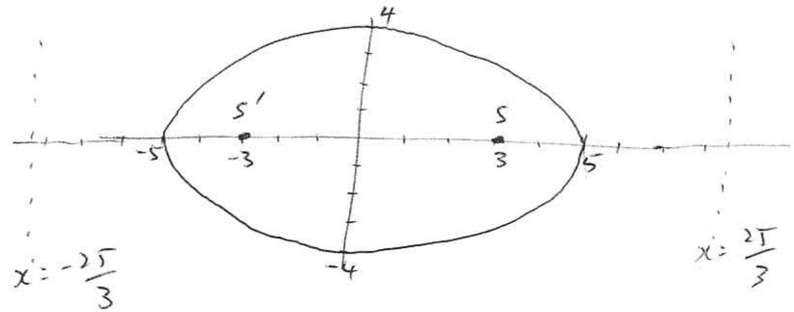
$b^2 = 25 - 9$ (from ΔOSb)
$= 16$
$\therefore b = 4$

Hence $\frac{x^2}{25} + \frac{y^2}{16} = 1$

(ii) Directrices $x = \pm \frac{a}{e}$ $\quad (e = \frac{3}{5})$

$\therefore x = \pm 5 \times \frac{5}{3}$
$= \pm \frac{25}{3}$

foci $(\pm 3, 0)$



Q5

(a) $u_1 = 1 \quad u_2 = 5 \quad u_n = 5u_{n-1} - 6u_{n-2}$

Prove $u_n = 3^n - 2^n$

For $n=3$: $u_3 = 5u_2 - 6u_1$
$= 5(5) - 6(1)$
$= 19$

1. Show true for $n=3$
$u_3 = 3^3 - 2^3$
$= 27 - 8 = 19$

∴ true for $n=3$

2. Assume true for $n=k$
i.e given $u_k = 5u_{k-1} - 6u_{k-2}$
then $u_k = 3^k - 2^k$

3. Prove true for $n=k+1$
i.e given $u_{k+1} = 5u_k - 6u_{k-1}$
show that $u_{k+1} = 3^{k+1} - 2^{k+1}$

LHS $= 5u_k - 6u_{k-1}$
$= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1})$
$= 5(3^k - 2^k) - [2 \cdot 3 \cdot 3^{k-1}] + [3 \cdot 2 \cdot 2^{k-1}]$
$= 5 \cdot 3^k - 5 \cdot 2^k - 2 \cdot 3^k + 3 \cdot 2^k$
$= 3 \cdot 3^k - 2 \cdot 2^k$
$= 3^{k+1} - 2^{k+1}$
$=$ RHS

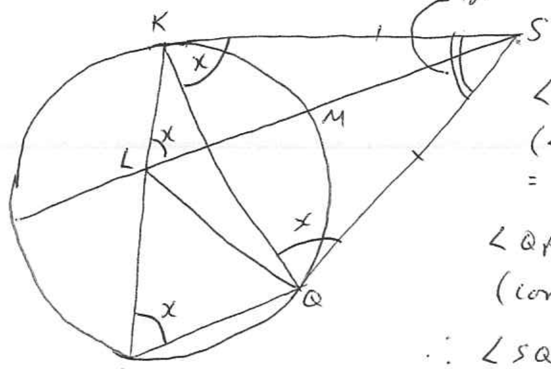
∴

(11)

(12)

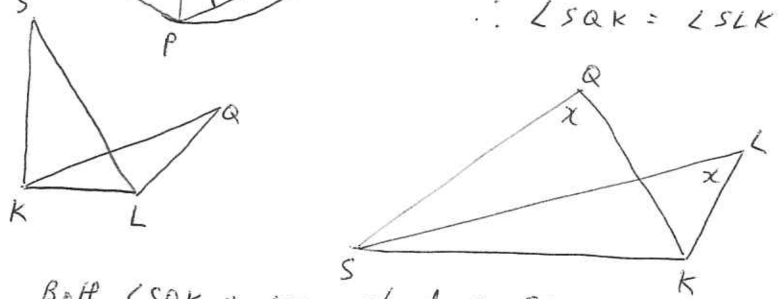
Q 5(b)

(i)



$\angle SQK = \angle QPK = x$
($\angle$ between tangent & chord = $\angle$ in alt segment).
$\angle QPK = \angle SLK$
(corresponding $\angle$'s in // lines =)
∴ $\angle SQK = \angle SLK$

(ii)



Both $\angle SQK$ & $\angle SLK$ stand on same arc KS of circle LKSQ

(iii) ΔKSQ is isosceles with $KS = QS$ ∴ $\angle SKQ = \angle SQK = x$
in ΔKSQ, $\angle KSQ = 180 - 2x$ ($\angle$ sum of Δ)
∴ in cyclic quad ($\angle KSQ$) $\angle KLQ = 2x$ (suppl with $\angle KSQ$).
now $\angle KLQ$ is ext $\angle$ of ΔPLQ.
∴ $\angle PQL = x$
∴ ΔPQL is isosceles
∴ $\angle Q = \angle P$ (sides opp = $\angle$'s).

5(c)

$$\sum_{r=0}^{n} (-1)^n \frac{{}^nC_r}{x+n} = \frac{n!}{x(x+1)(x+2)\cdots(x+n)}$$

if $x=1$:

$$\sum_{r=0}^{n} (-1)^n \frac{{}^nC_r}{1+n} = \frac{n!}{1(2)(3)\cdots(n+1)}$$

$$\text{RHS} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

$$\text{LHS} = \frac{(-1)^0\,{}^nC_0}{1} + \frac{(-1)^1\,{}^nC_1}{2} + \frac{(-1)^2\,{}^nC_2}{3} + \dots + \frac{(-1)^n\,{}^nC_n}{1+n}$$

$$= 1 + \frac{-{}^nC_1}{2} + \frac{{}^nC_2}{3} + \dots + \frac{(-1)^n}{1+n}$$

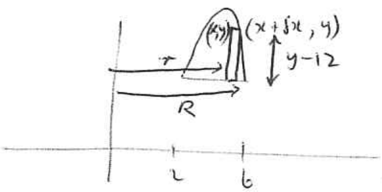
i.e $1 - \frac{1}{2}{}^nC_1 + \frac{1}{3}{}^nC_2 - \dots \frac{(-1)^n}{1+n} = \frac{1}{n+1}$

(d) $12 = 8x - x^2$

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0$

POI: $(2,12)$ & $(6,12)$



X-sect Area of shell

$= \pi(R^2 - r^2)$

$= \pi((x+\delta x)^2 - x^2)$

$= \pi(x^2 + 2x\delta x + \delta x^2 - x^2)$

$= \pi\, 2x\delta x$ (ignoring δx^2)

$= 2\pi x \delta x$

Vol of typical shell

$\delta V = 2\pi x \delta x (y - 12)$

$\therefore$ total volume

$$= 2\pi \int_2^6 x(8x - x^2 - 12)\,dx$$

$$= 2\pi \int_2^6 8x^2 - x^3 - 12x \,dx$$

$$= 2\pi \left[\frac{8x^3}{3} - \frac{x^4}{4} - 6x^2\right]_2^6$$

$$= 2\pi \left[\left(\frac{8\times 6^3}{3} - \frac{6^4}{4} - 6\times 36\right) - \left(\frac{8\times 2^3}{3} - \frac{2^4}{4} - 6\times 4\right)\right]$$

Q6

a(i) $$\frac{A(\pi - 2x) + Bx}{x(\pi - 2x)} = \frac{1}{x(\pi - 2x)}$$

$$\tfrac{1}{\pi}(\pi - 2x) + Bx = 1$$

$$1 - \frac{2x}{\pi} + Bx = 1$$

$$Bx = \frac{2x}{\pi}$$

$$\therefore B = \frac{2}{\pi}$$

(ii) $$\int_{\pi/6}^{\pi/3} \frac{1}{\pi}\frac{1}{x} + \frac{1}{\pi}\left(\frac{2}{\pi - 2x}\right)$$

$$= \frac{1}{\pi}\ln x - \frac{1}{\pi}\ln(\pi - 2x)$$

$$= \frac{1}{\pi}\left[\ln x - \ln(\pi - 2x)\right]$$

$$= \frac{1}{\pi}\left[\ln\frac{x}{\pi - 2x}\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{\pi}\left[\ln\frac{\pi/3}{\pi/3} - \ln\frac{\pi/6}{4\pi/6}\right]$$

$$= \frac{1}{\pi}\left[0 - \ln\frac{1}{4}\right]$$

$$= \frac{1}{\pi}\left[-\ln 2^{-2}\right]$$

$$= \frac{1}{\pi}\left[2\ln 2\right]$$

$$= \frac{2}{\pi}\ln 2$$

(iii) $\int_a^b f(x)\,dx$ Let $u = a+b-x \Rightarrow du = -dx$

$x = a+b-u \quad \therefore dx = -du$

when $x = a$, $u = b$

$x = b$, $u = a$

$$\therefore \int_a^b f(x)\,dx = \int_b^a f(a+b-u)\,-du$$

$$= \int_a^b f(a+b-u)\,du$$

$$= \int_a^b f(a+b-x)\,dx$$

iv) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x\,dx}{x(\pi-2x)} = \int \frac{\cos^2\left[\frac{\pi}{6}+\frac{\pi}{3}-x\right]\,dx}{\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\left[\pi - 2\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\right]}$$

$$= \int \frac{\cos^2\left(\frac{\pi}{2}-x\right)\,dx}{\left(\frac{\pi}{2}-x\right)\left[\pi-2\left(\frac{\pi}{2}-x\right)\right]}$$

$$= \int \frac{\sin^2 x}{\left(\frac{\pi}{2}-x\right)(2x)}\,dx$$

$$= \int \frac{1-\cos^2 x}{x(\pi-2x)}$$

So, $$\int \frac{\cos^2 x\,dx}{x(\pi-2x)} = \int \frac{1}{x(\pi-2x)} - \frac{\cos^2 x}{x(\pi-2x)}\,dx$$

$$\therefore 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x\,dx}{x(\pi-2x)} = \frac{2}{\pi}\ln 2$$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x\,dx}{x(\pi-2x)} = \frac{1}{\pi}\ln 2$$

6 (b)(i) $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$ & $y^2 = \frac{a^2}{4}\left[e^{2x/a} + 2 + e^{-2x/a}\right]$

$$\frac{dy}{dx} = \frac{a}{2}\left[\frac{1}{a}e^{x/a} - \frac{1}{a}e^{-x/a}\right]$$

$$= \frac{1}{2}\left[e^{x/a} - e^{-x/a}\right]$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left[e^{2x/a} - 2 + e^{-2x/a}\right]$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{e^{2x/a}}{4} - \frac{1}{2} + \frac{e^{-2x/a}}{4}$$

$$= \frac{1}{4}\left[e^{2x/a} + 2 + e^{-2x/a}\right]$$

$$= \frac{y^2}{a^2}$$

(ii) $$S = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^x \sqrt{\frac{y^2}{a^2}}\,dx$$

$$= \int_0^x \frac{y}{a}\,dx$$

$$= \frac{1}{2}\int_0^x e^{x/a} + e^{-x/a}\,dx$$

$$= \frac{1}{2}\left[a e^{x/a} - a e^{-x/a}\right]_0^x$$

$$= \frac{a}{2}\left[e^{x/a} - e^{-x/a}\right] - \frac{a}{2}[a-a]$$

$$\therefore S = \frac{a}{2}\left[e^{x/a} - e^{-x/a}\right]$$

and $$y^2 - a^2 = \frac{a^2}{4}\left[e^{x/a} + e^{-x/a}\right]^2 - a^2$$

$$= \frac{a^2}{4}\left[e^{2x/a} + 2 + e^{-2x/a} - 4\right]$$

$$= \frac{a^2}{4}\left[e^{2x/a} - 2 + e^{-2x/a}\right]$$

$$= \frac{a^2}{4}\left(e^{x/a} - e^{-x/a}\right)^2$$

$$= S^2$$

$$\therefore S = \sqrt{y^2 - a^2}$$

Q7

a(i) $f(x) = \sqrt{2-\sqrt{x}}$ $\quad x \geq 0$ for $\sqrt{x}$

$x \leq 4$ for $\sqrt{2-\sqrt{4}}$

(ii) $f(x) = \left(2 - x^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}x^{-\frac{1}{2}}\right)\left(2-x^{\frac{1}{2}}\right)^{-\frac{1}{2}}$

$= \dfrac{-1}{4\sqrt{x}\,\sqrt{2-\sqrt{x}}} < 0$ for $0 \leq x \leq 4$

$\therefore$ a decreasing function.

$x = 0 \quad f(x) = \sqrt{2}$

$x = 4 \quad f(x) = 0$

Range $0 \leq y \leq \sqrt{2}$

(iii) $u = 2 - \sqrt{x}$

$du = -\frac{1}{2}x^{-\frac{1}{2}}\,dx$

$= \dfrac{-1}{2\sqrt{x}}\,dx$

$\therefore dx = -2\sqrt{x}\,du$

$= -2(2-u)\,du$

when $x = 4$, $u = 0$

$x = 0$, $u = 2$

$\therefore \int_0^4 \sqrt{2-\sqrt{x}}\,dx$

$= -2\int_2^0 \sqrt{u}\,(2-u)\,du$

$= 2\int_0^2 2u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$

$= 2\left[\dfrac{4u^{\frac{3}{2}}}{3} - \dfrac{2u^{\frac{5}{2}}}{5}\right]_0^2$

$= 2\left[\left(\frac{4}{3}2^{\frac{3}{2}} - \frac{2}{5}2^{\frac{5}{2}}\right) - (0-0)\right]$

$= 2\left(\frac{8}{3}\sqrt{2} - \frac{8}{5}\sqrt{2}\right) = \dfrac{32\sqrt{2}}{15}$

7(b) $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$

$= \int_0^{\frac{\pi}{4}} \tan^2 x \tan^{n-2} x\,dx$

$= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\tan^{n-2} x\,dx$

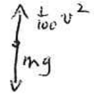
$= \int \underbrace{\sec^2 x}_{f'(x)}\, \underbrace{\tan^{n-2} x}_{f(x)}\,dx - \int \tan^{n-2} x\,dx$

$= \left[\dfrac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$

$I_n = \left(\frac{1}{n-1} - 0\right) - I_{n-2}$

$\therefore I_n + I_{n-2} = \dfrac{1}{n-1}$

(c)(i)



$F = m\ddot{x} = mg - \frac{1}{100}v^2 \qquad m = 1$

$\therefore \ddot{x} = g - \frac{1}{100}v^2$

(ii) V_T occurs when $\ddot{x} = 0$

i.e. $0 = g - \dfrac{V_T^2}{100}$

$V_T^2 = 100g$

$V_T = 10\sqrt{g}$

(iii) $\ddot{x} = g - \dfrac{v^2}{100}$ $\quad$ we want $v = f(x)$

$v\dfrac{dv}{dx} = g - \dfrac{v^2}{100}$

$= \dfrac{100g - v^2}{100}$

$\therefore \dfrac{dv}{dx} = \dfrac{100g - v^2}{100v} \Rightarrow \dfrac{dx}{dv} = \dfrac{100v}{100g - v^2}$

$x = \int_0^v \dfrac{100v}{100g - v^2}\,dv$

$x = -50\int_0^v \dfrac{-2v}{100g - v^2}\,dv$

$= \left[-50\ln|100g - v^2|\right]_0^v$

$\therefore x = -50\ln(100g - v^2) + 50\ln(100g)$ $\quad$ now write $v = f(x)$

$= 50\ln\dfrac{100g}{100g - v^2}$

$\dfrac{x}{50} = \ln\dfrac{100g}{100g - v^2}$

$e^{\frac{x}{50}} = \dfrac{100g}{100g - v^2}$

$\dfrac{100g - v^2}{100g} = e^{-\frac{x}{50}}$

$v^2 = 100g - 100g\,e^{-\frac{x}{50}}$

$= 100g\left(1 - e^{-\frac{x}{50}}\right)$

$= V_T^2\left(1 - e^{-\frac{x}{50}}\right)$

8(a)(i) $x^2 + 2xy + y^5 = 4$

$2x + 2y + \dfrac{dy}{dx}2x + 5y^4\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}\left[2x + 5y^4\right] = -2x - 2y$

$\therefore \dfrac{dy}{dx} = \dfrac{-2x - 2y}{2x + 5y^4}$

$\therefore$ @ $P(X, Y)$, $\dfrac{dy}{dx} = \dfrac{-2X - 2Y}{2X + 5Y^4}$

(8) (b) (i) $\tan 4\theta = 1$ $\qquad 0 \le \theta \le \pi$

$0 \le 4\theta \le 4\pi$

$4\theta = \frac{\pi}{4},\ \pi + \frac{\pi}{4},\ 2\pi + \frac{\pi}{4},\ 3\pi + \frac{5\pi}{4}$

$4\theta = \frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{9\pi}{4},\ \frac{13\pi}{4}$

$\theta = \frac{\pi}{16},\ \frac{5\pi}{16},\ \frac{9\pi}{16},\ \frac{13\pi}{16}$

(ii) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

(iii) Let $t = \tan\theta$ $\qquad \tan 4\theta = \frac{2\tan 2\theta}{1-\tan^2 2\theta}$

$$= \frac{2\left(\frac{2t}{1-t^2}\right)}{1-\left(\frac{2t}{1-t^2}\right)^2} \times \frac{(1-t^2)^2}{(1-t^2)^2}$$

$$= \frac{4t(1-t^2)}{(1-t^2)^2 - 4t^2} = \frac{4t-4t^3}{1-2t^2+t^4-4t^2}$$

$\therefore \tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$

(iv) Let $\tan 4\theta = 1$ and $\tan\theta = x$

$1 = \frac{4x-4x^3}{1-6x^2+x^4}$

$x^4 - 6x^2 + 1 = 4x - 4x^3$

$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

solns are $x = \tan\theta$ where θ is soln to $\tan 4\theta = 1$

$\therefore$ solns are $\tan\frac{\pi}{16},\ \tan\frac{5\pi}{16},\ \tan\frac{9\pi}{16},\ \tan\frac{13\pi}{16}$.

(v) Product of roots $= +\frac{e}{a}$ in $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$= 1$

(vi) $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$x^2\left(x^2 + 4x - 6 - \frac{4}{x} + \frac{1}{x^2}\right) = 0$

$x^2 \neq 0$ $\quad \left(\left(x^2 + \frac{1}{x^2}\right) + 4\left(x - \frac{1}{x}\right) - 6\right) = 0$

$\left(\left(x - \frac{1}{x}\right)^2 + 2 + 4\left(x - \frac{1}{x}\right) - 6\right) = 0$

$\left(\left(x - \frac{1}{x}\right)^2 + 4\left(x - \frac{1}{x}\right) - 4\right) = 0$

Let $u = x + \frac{1}{x}$ $\qquad u^2 + 4u - 4 = 0$

$u = \frac{-4 \pm \sqrt{16 + 4\times 4}}{2} = \frac{-4 \pm 4\sqrt{2}}{2} = -2 \pm 2\sqrt{2}$

$x - \frac{1}{x} = -2 \pm 2\sqrt{2}$

smallest value is $\tan\frac{\pi}{16} - \frac{1}{\tan\frac{\pi}{16}} = -2 - 2\sqrt{2}$

$\therefore \tan\frac{\pi}{16} - \cot\frac{\pi}{16} = -2 - 2\sqrt{2}$

(ii) horiz @ P suggests $-2X - 2y = 0$

$X = -y$

i.e $X^2 + 2X(-X) + (-X)^5 = 4$

$X^2 - 2X^2 - X^5 - 4 = 0$

i.e $X^5 + X^2 + 4 = 0$

(iii) Let $y = X^5 + X^2 + 4$

$\frac{dy}{dx} = 5X^4 + 2X$

Let $\frac{dy}{dx} = 0$ $\qquad 2X + 5X^4 = 0$

$X(2 + 5X^3) = 0$

$X = 0$ or $X^3 = \frac{-2}{5}$

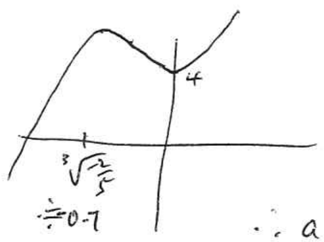
$X = \sqrt[3]{\frac{-2}{5}}$

$\frac{d^2y}{dx^2} = 20X^3 + 2$

@ $X = 0$ $\frac{d^2y}{dx^2} = 2 > 0$ $\therefore$ ∪ min.

at $x = \sqrt[3]{\frac{-2}{5}}$ $\quad \frac{d^2y}{dx^2} = 20\left(\sqrt[3]{\frac{-2}{5}}\right)^3 + 2 < 0$ ∩ max

$\therefore$ min at $(0,4)$ & max at $x = \sqrt[3]{\frac{-2}{5}}$



when $x = -1$, $y = -1 + 1 + 4 > 0$

$x = -2$, $y = -32 + 4 + 4 < 0$

$\therefore$ a unique real solution occurs between -1 & -2.